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Deformations of the Riccati equation by using Miura-type transformations

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Abstract. Using some different Miura-type transformations, a C-integrable ordinary differential equation, the Riccati equation, is deformed to some different S-integrable models such as the $(1+1)$-dimensional and $(2+1)$-dimensional sinh–Gordon equations and Mikhailov–Dodd–Bullough equations.

1. Introduction

In recent years, the study of both the S-integrable (integrable via an appropriate spectral transform) and C-integrable (solvable via an appropriate change of variables) models has attracted much attention from mathematicians and physicists [1, 2]. To solve a nonlinear problem, the first two things one wishes to know are: (i) Whether the problem can be changed to a linear one directly by using some suitable transformations? (ii) Whether a higher-dimensional problem can be (partially) solved by means of some lower-dimensional ones? For the first problem, many physically significant equations are solved satisfactorily. Some C-integrable models have been directly changed to linear ones, for example the Burgers equation and the Liouville equation have been changed to the linear heat conductive equation and the linear wave equation, respectively, by means of the Cole–Hopf transformation. Many S-integrable models, such as the Korteweg–de Vries (KdV), Kadomtsev–Petviashvili (KP), sine–Gordon (sG) and nonlinear Schrödinger (NLS) equations, are solved with the help of some linear spectral problems. The second problem has been solved partially by using the symmetry constraint (or reduction) methods. The classical and non-classical Lie symmetry reduction approach has been used to reduce both integrable and non-integrable models [3–5]. Using the generalized local and non-local symmetry constraint method (nonlinearization approach of the Lax pairs), many higher-dimensional *integrable* models can be solved by some lower-dimensional integrable systems [6–8].

In this paper, we are interested in two related problems: (i) Can we change a C-integrable model to an S-integrable model? (ii) May a lower-dimensional integrable model be deformed to those of higher dimensions? Fortunately, the answer to these two problems is also positive. In section 2 of this paper, a brief report of the general theory related to

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transformations between two equations is sketched. In section 3, we deform the following one-dimensional Riccati equation

$$\phi_t = \phi^2 \tag{1}$$

(which is C-integrable of course) to some $(1+1)$- and $(2+1)$-dimensional sinh–Gordon (ShG) type equations. Section 4 is devoted to the study of the deformation of the same equation (1) to the $(1+1)$-dimensional and (2+1)-dimensional Mikhailov–Dodd–Bullough (MDB) equations. Section 5 is a summary and discussion.

2. General theory

If the following lower-dimensional evolution equation

$$\phi_t = K(\phi) \tag{2}$$

is integrable (we may suppose further that it is C-integrable), we may change it to some other equations by using some complicated transformations, for example

$$B(\phi, v) = 0 \qquad \text{or} \qquad \phi = \varphi(v). \tag{3}$$

Using the transformation relation (3) and the ϕ evolution equation (2), the related v equation can be obtained in the following way: differentiating the transformation relation (3) with respect to t at first, we have

$$B'_\phi \phi_t + B'_v v_t = 0 \qquad \text{or} \qquad \phi_t = \varphi'_v v_t \tag{4}$$

where B'_ϕ, B'_v and φ'_v are partial linearized operators of B and φ, say

$$B'_\phi f = \lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\epsilon} B(\phi + \epsilon f) \tag{5}$$

for arbitrary f. Now substituting (2)–(4) yields the v evolution equation

$$v_t = -B'^{-1}_v B'_\phi K(\phi)|_{\phi=\varphi(v)} \qquad \text{or} \qquad v_t = \varphi'^{-1}_v K(\phi)|_{\phi=\varphi(v)} \tag{6}$$

where B'^{-1}_v and φ'^{-1}_v are the inverse operators of B'_v and φ'_v.

The answer to the question (ii) can be verified from the above general discussion, because if the dimension of the transformation relation (3) is higher than that of the original evolution equation (2), the result equation must possess higher dimensions. Furthermore, it is also possible to answer question (i) from the above discussion: if some differential operators are included in the transformation relation (3), some integral operators must be included in the inverse operator B'^{-1}_v and the result equation may be S-integrable even if the original equation is C-integrable. To see the conclusion more clearly, we prefer to turn to some concrete examples in the following sections.

3. ShG extensions deformed from the Riccati equation

3.1. *(1 + 1)-dimensional ShG equation*

It is well known that to solve the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0 \tag{7}$$

the Miura transformation (MT) [9]

$$u = -\tfrac{1}{2} v_{xx} - \tfrac{1}{4} v_x^2 \tag{8}$$

plays an important role. The MT (8) changes the solutions of the potential mKdV equation

$$v_t + v_{xxx} - 2v_x^3 = 0 \tag{9}$$

to those of the KdV equation (7). Taking a Cole–Hopf transformation (CHT) $v_x = (\ln\psi)_x$ (8) leads to the linear spectral problem of the KdV equation (the spectral parameter can be added into the problem simply by replacing u by $u - \lambda$ thanks to the Galileo invariance of the KdV equation). On the other hand, if we take an alternative transformation

$$v = \ln\phi_x \tag{10}$$

(8) becomes

$$u = -\frac{1}{2}\frac{\phi_{xxx}}{\phi_x} + \frac{1}{4}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{11}$$

and the KdV equation (7) is transformed to its Schwarz form [10]

$$\frac{\phi_t}{\phi_x} + \{\phi; x\} = 0 \tag{12}$$

where the Schwarz derivative $\{\phi; x\}$ is defined as

$$\{\phi; x\} = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2. \tag{13}$$

Cancelling u from (8) and (11), we get a suitable candidate for deforming one integrable model to another

$$\frac{\phi_{xxx}}{\phi_x} - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 - v_{xx} - \frac{1}{2}v_x^2 \equiv B_1(\phi, v) = 0. \tag{14}$$

Now we apply the MT related transformation (14) to the Riccati equation (1). Differentiating (14) with respect to t and using the Riccati equation (1), we get

$$(\partial_x + v_x)v_{xt} = 4\phi_{xx}. \tag{15}$$

From (14), one can easily express ϕ_x by v; the result reads

$$\phi_x = \mathrm{e}^v\left(C_2 + C_1\int^x \mathrm{e}^{-v(x')}\,\mathrm{d}x'\right)^2 \tag{16}$$

where C_1 and C_2 are two arbitrary functions of t. Substituting (16) into (15) and solving for v_{xt} leads to the first type of deformed equation of the Riccati equation

$$\begin{aligned} v_{xt} = {} & 2C_2^2\,\mathrm{e}^v - C_3\,\mathrm{e}^{-v} + 4\,\mathrm{e}^{-v}C_1\bigg[C_2\left(\int^x \mathrm{e}^{v(x')}\,\mathrm{d}x' + \mathrm{e}^{2v}\int^x \mathrm{e}^{-v(x')}\,\mathrm{d}x'\right) \\ & + C_1\left(\int^x \mathrm{e}^{v(x_1)}\int^{x_1} \mathrm{e}^{-v(x')}\,\mathrm{d}x'\,\mathrm{d}x_1 + \mathrm{e}^{2v}\int^x \mathrm{e}^{-v(x_1)}\int^{x_1} \mathrm{e}^{-v(x')}\,\mathrm{d}x'\,\mathrm{d}x_1\right)\bigg] \end{aligned} \tag{17}$$

with C_3 being a further arbitrary function of t. Taking the following further transformations

$$w = v + \ln\left(C_2\sqrt{\frac{2}{C_3}}\right) \tag{18}$$

$$\tau = \int^t \sqrt{2C_3(t')}C_2(t')\,\mathrm{d}t' \tag{19}$$

(17) can be recast in the form

$$w_{x\tau} = \sinh w + A(\tau)\left(e^{-w}\int^x e^{w(x')}\,dx' + e^{w}\int^x e^{-w(x')}\,dx'\right)$$
$$+B(\tau)\left(e^{-w}\int^x e^{w(x_1)}\int^{x_1} e^{-w(x')}\,dx'\,dx_1 + e^{w}\int^x e^{-w(x_1)}\int^{x_1} e^{-w(x')}\,dx'\,dx_1\right) \tag{20}$$

with

$$A(\tau) = 2\sqrt{\frac{2}{C_3}}C_1\bigg|_{t=\tau^{-1}(\tau)} \qquad B(\tau) = 4\frac{C_1^2}{C_3}\bigg|_{t=\tau^{-1}(\tau)}. \tag{21}$$

When we take the arbitrary integral function $C_1 = 0$ ($A(\tau) = B(\tau) = 0$), (20) is the well known ShG equation. The integrability of (20) is obvious not only because it is obtained from the deformation of the Riccati equation (1) but also because (20) is really a non-local flow equation of the potential mKdV equation. It is known that [11]

$$K_{-1}^{(1)} \equiv \int^x \sinh w(x')\,dx' \tag{22}$$

$$K_{-1}^{(2)} \equiv \int^x \left(e^{-w(x_1)}\int^{x_1} e^{w(x')}\,dx' + e^{w(x_1)}\int^{x_1} e^{-w(x')}\,dx'\right)dx_1 \tag{23}$$

$$K_{-1}^{(3)} \equiv \int^x \left(e^{-w(x_2)}\int^{x_2} e^{w(x_1)}\int^{x_1} e^{-w(x')}\,dx'\,dx_1 + e^{w(x_2)}\int^{x_2} e^{-w(x_1)}\int^{x_1} e^{-w(x')}\,dx'\,dx_1\right)dx_2 \tag{24}$$

are all symmetries of the potential mKdV equation with field w.

If we want only to obtain the ShG equation from the Riccati equation, it is enough to select a special case for the integral functions

$$C_1 = 0 \qquad C_2^2 = \tfrac{1}{2} \qquad C_3 = 1$$

in (17).

It is also known that the usual ShG equation is S-integrable [2] rather than C-integrable. The reasons that one can deform a C-integrable model (1) to S-integrable ones (say, ShG equation) are: (i) there are some possible C-integrable reductions of an S-integrable model; (ii) the deformation relations (say, equation (14)) may be non-invertable if some differential operators are included.

3.2. (2 + 1)-dimensional ShG equation

There are different types of extensions of the Miura transformation in high-dimensions. Using these different types of Miura transformations to deform the Riccati equation (1), we can obtain different types of ShG extensions. Here we write down only one special example.

One of the simple significant (2+1)-dimensional extensions of the Miura transformation may have the form

$$u = -\frac{1}{2}v_{xx} - \frac{1}{4}v_x^2 - e^{-2v}\left(\int^x v_y\,e^{v}\,dx'\right)^2 + \frac{1}{2}\int^x \left(e^{-v}\int_1^x v_y\,e^{v}\,dx'\right)_y dx_1. \tag{25}$$

If using transformation (8) on the above equation, we have

$$u = -\frac{1}{2}\frac{\phi_{xxx}}{\phi_x} + \frac{1}{4}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 - \frac{\phi_y^2}{\phi_x^2} + \frac{1}{2}\int^x \left(\frac{\phi_y}{\phi_x}\right)_y \mathrm{d}x'. \tag{26}$$

By means of the standard singularity analysis approach [10], one can prove that the transformation (26) transforms the KP equation to its Schwarts form (up to a transformation kernel).

As previously shown, cancelling u from (25) and (26) yields an appropriate deformation relation

$$-v_{xx} - \frac{1}{2}v_x^2 - 2\,\mathrm{e}^{-2v}\left(\int^x v_y\,\mathrm{e}^v\,\mathrm{d}x'\right)^2 + \int^x \left(\mathrm{e}^{-v}\int^{x_1} v_y\,\mathrm{e}^v\,\mathrm{d}x'\right)_y \mathrm{d}x_1$$
$$+\frac{\phi_{xxx}}{\phi_x} - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 + 2\frac{\phi_y^2}{\phi_x^2} - \int^x \left(\frac{\phi_y}{\phi_x}\right)_y \mathrm{d}x' = 0 \tag{27}$$

for the Riccati equation. It is obvious that when v (and then ϕ) is y-independent, equations (25), (26) and (27) are reduced back to (8), (11) and (14), respectively.

Now we can use the $(2+1)$-dimensional transformation relation (27) to deform the same Riccati equation (1) to a $(2+1)$-dimensional sinh–Gordon equation.

In order to write the final result in differential form we introduce two auxiliary fields $\psi(x,\ t)$ and $r(x,\ t)$ as

$$\psi_x = \frac{\phi_y}{\phi_x} \tag{28}$$

$$r_{xx} + r_x v_x = v_y \qquad \left(\text{i.e. } r_x = \mathrm{e}^{-v}\int^x v_y\,\mathrm{e}^v\,\mathrm{d}x'\right). \tag{29}$$

Substituting (28) and (29) into (27) yields

$$-v_{xx} - \frac{1}{2}v_x^2 - 2r_x^2 + r_y + \frac{\phi_{xxx}}{\phi_x} - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 + 2\psi_x^2 - \psi_y = 0. \tag{30}$$

As in the derivation of the $(1+1)$-dimensional ShG equation, differentiating (27) (or (30)) with respect to time t and using the Riccati equation (1) yields

$$(\partial_x + v_x)(v_{xt} - \tfrac{1}{2}C_1\,\mathrm{e}^{-v}) - 4(r_x r_{xt} - \psi_x\psi_{xt}) + r_{yt} - \psi_{yt} + 4\phi_{xx} \tag{31}$$

where a term with arbitrary function $C_1 \equiv C_1(y,\ t)$ has been added because $(\partial_x + v_x)\,\mathrm{e}^{-v} \equiv 0$. Equation (31) along with (28)–(30) is really a $(2+1)$-dimensional ShG extension.

For clarification, we rewrite (28)–(31) in an alternative form

$$\begin{cases} (\partial_x + v_x)(v_{xt} - C_1\sinh v) - 4(r_x r_{xt} - \psi_x\psi_{xt}) + r_{yt} - \psi_{yt} \\ \quad +(4v_x h^2 + 8hh_x + C_1 v_x)\,\mathrm{e}^v = 0 \\ r_{xx} + r_x v_x - v_y = 0 \\ \psi_{xx} + \psi_x(v + 2\ln h)_x - (v + 2\ln h)_y = 0 \\ h_{xx} + h_x v_x - hr_x^2 + \frac{1}{2}hr_y + h\psi_x^2 - \frac{1}{2}h\psi_y = 0 \end{cases} \tag{32}$$

by using the transformation

$$\phi_x = h^2\,\mathrm{e}^v. \tag{33}$$

It is clear that when the fields are y-independent, $r = \psi = 0$, $h = h(t) = \sqrt{C_1}/2$, the $(2+1)$-dimensional ShG equation (32) reduces back to the $(1+1)$-dimensional ShG equation.

As in the $(1+1)$-dimensional case, the $(2+1)$-dimensional ShG equation (32) obtained from the deformation of the Riccati equation is also S-integrable and it is a variant form of the negative KP equation [12].

4. MDB extensions deformed from the Riccati equation

4.1. (1 + 1)-dimensional MDB equation

In the (1 + 1)-dimensional case, in addition to the SG (or ShG) and Liouville equations, there is another Klein–Gordon-type integrable model, the MDB equation, with spacetime symmetric form. It is interesting that it can also be obtained from the deformation of the Riccati equation.

Now we take a similar deformation relation as (14)

$$\frac{\phi_{xxx}}{\phi_x} - v_{xx} - v_x^2 \equiv B_2(\phi, \ v) = 0. \tag{34}$$

The only difference between (34) and (14) is some constant coefficients of the transformation relations.

Differentiating (34) with respect to t and using the Riccati equation (1) again, we obtain

$$(\partial_x + 2v_x)v_{xt} = 6\phi_{xx}. \tag{35}$$

The general solution of (34) now reads

$$\phi_x = \mathrm{e}^v\left(C_2 + C_1\int^x \mathrm{e}^{-2v(x')}\,\mathrm{d}x'\right) \tag{36}$$

with C_1 and C_2 being two arbitrary functions of t. Substituting (36) into (35) and solving for v_{xt} leads to a MDB-type equation

$$v_{xt} = C_3\,\mathrm{e}^{-2v} + 2C_2\,\mathrm{e}^v + 2C_1\left(\mathrm{e}^v\int^x \mathrm{e}^{-2v(x')}\,\mathrm{d}x' + 2\,\mathrm{e}^{-2v}\int^x \mathrm{e}^{v(x')}\,\mathrm{d}x'\right) \tag{37}$$

with C_3 being a further arbitrary function of t.

When we take $C_1 = 0$, (37) becomes the usual MDB equation

$$w_{x\tau} = \mathrm{e}^w + \mathrm{e}^{-2w} \tag{38}$$

with

$$\tau = \int^t (4C_2^2C_3)^{1/3} \qquad w = v + \frac{1}{3}\ln\left(\frac{2C_2}{C_3}\right). \tag{39}$$

The integrability of (37) is also obvious, not only because it is obtained from the deformation of the Riccati equation (1), but also because (37) is really a non-local flow equation of the potential Sawada–Kortera (SK) equation. Using the results of [13], it is easy to prove that

$$K_{-1}^{(1)} \equiv \int^x \mathrm{e}^{w(x_1)}\,\mathrm{d}x_1 \tag{40}$$

$$K_{-1}^{(2)} \equiv \int^x \mathrm{e}^{-2w(x_1)}\,\mathrm{d}x_1 \tag{41}$$

and

$$K_{-1}^{(3)} \equiv \int^x \left(\mathrm{e}^{w(x_1)}\int^{x_1} \mathrm{e}^{-2w(x')}\,\mathrm{d}x' + 2\,\mathrm{e}^{-2w(x_1)}\int^{x_1} \mathrm{e}^{w(x')}\,\mathrm{d}x'\right)\mathrm{d}x_1 \tag{42}$$

are all non-local symmetries of the potential SK equation with field w.

4.2. (2 + 1)-dimensional MDB equation

To get a (2+1)-dimensional MDB extension, we can make use of the following deformation relation:

$$v_{xx} + v_x^2 + \mathrm{e}^{-v}\int^x v_y\,\mathrm{e}^{v}\,\mathrm{d}x' - \frac{\phi_y}{\phi_x} - \frac{\phi_{xxx}}{\phi_x} = 0. \tag{43}$$

When the fields v and ϕ are y-independent, (43) is reduced back to (34).

Differentiating (43) with respect to time t and using the Riccati equation (1) again leads to

$$(\partial_x + 2v_x)v_{xt} = 6\phi_{xx} + v_t\,\mathrm{e}^{-v}\int^x v_y\,\mathrm{e}^{v}\,\mathrm{d}x' - \mathrm{e}^{-v}\int^x (\mathrm{e}^{v})_{yt}\;\mathrm{d}x'. \tag{44}$$

In order to compare the result with that in the $(1+1)$-dimensional case, we rewrite (43) and (44) as

$$\begin{cases} (\partial_x + 2v_x)(v_{xt} - C_3\,\mathrm{e}^{-2v} - C_2\,\mathrm{e}^{v}) \\ \quad = 6(h_x + hv_x)\,\mathrm{e}^{v} + v_t\,\mathrm{e}^{-v}\displaystyle\int^x v_y\,\mathrm{e}^{v}\,\mathrm{d}x' - \mathrm{e}^{-v}\int^x (\mathrm{e}^{v})_{yt}\;\mathrm{d}x' - 3C_2\,\mathrm{e}^{v}v_x \\ h_{xx} + 2v_xh_x - \mathrm{e}^{-v}\displaystyle\int^x \left(h_{x_1}\int^{x_1}\mathrm{e}^{v}\,\mathrm{d}x'\right)_y\,\mathrm{d}x_1 = 0 \end{cases} \tag{45}$$

by means of the transformation

$$\phi_x = h\,\mathrm{e}^{v} \tag{46}$$

where C_2 and C_3 are arbitrary functions of $\{y,\ t\}$. When the fields are y-independent, $\psi = 0$, $C_2 = C_2(t)$, $C_3 = C_3(t)$, $h = C_2/2$, the $(2+1)$-dimensional MDB equation (45) reduces back to the $(1+1)$-dimensional MDB-type equation (37).

5. Summary and discussion

Usually, it is quite easy to reduce a complicated theory to a simple one, for example any quantum theory will reduce back to a classical one by ignoring the Plank constant $\hbar$, and any relativistic theory will be reduced back to a corresponding non-relativistic theory by taking the light velocity to infinity. However, the inverse procedure, i.e. the deformation of a simple theory to a complex one, is very difficult. Fortunately, it is possible to accomplish the inverse procedure in some special cases. For instance, the classical Yang–Baxter equation may evolve into the quantum Yang–Baxter equation, some critical phenomenon theory treated by conformal field theory, where the mass is zero, may be deformed to a theory for the non-zero mass case [14] and some types of special solutions of the single sine–Gordon equation may be deformed to that of the double sine–Gordon equation [15].

In this paper, we see that the deformation idea can also be used to obtain higher-dimensional integrable models from lower-dimensional ones. Using some Miura-type transformations to a trivial C-integrable model, the Riccati equation $\phi_t = \phi^2$, we have found some generalized $(1+1)$-dimensional and $(2+1)$-dimensional S-integrable models. The usual $(1+1)$-dimensional sinh–Gordon and the $(1+1)$-dimensional Mikhailov–Dodd–Boullough equations are just special cases and these equations have been extended to $(2+1)$ dimensions.

Reducing a higher-dimensional S-integrable model to a lower-dimensional one, one may obtain a lower-dimensional C-integrable model. Therefore, it is possible to get an S-integrable model from a suitable deformation of a lower-dimensional C-integrable model.

From many different methods (such as the classical Lie approach) which can be used to reduce many higher-dimensional integrable models to ordinary different equations, one always obtain some trivial C-integrable models or six Painlevé reductions. So we believe that any higher-dimensional integrable models can be obtained from several lower-dimensional integrable models by using some suitable deformation relations. In particular, one can use this idea to obtain $(3+1)$-dimensional integrable models (if they exist). Further study on this topic continues.

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References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons Nonlinear Evolution Equations and Inverse Scattering (London Mathematical Society Lecture Note Series 149)* (Cambridge: Cambridge University Press)
[2] Calogero F 1989 *J. Math. Phys.* **30** 28
[3] Bluman G W and Kumei S 1989 Symmetries and differential equations *Appl. Math. Sci.* vol 81 (Berlin: Springer)
Olver P J 1993 *Applications of Lie Groups to Differential Equations (Graduate Texts Math. 107)* 2nd edn (New York: Springer)
[4] Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201
Clarkson P A 1989 *J. Phys. A: Math. Gen.* **22** 2355
Clarkson P A 1989 *J. Phys. A: Math. Gen.* **22** 3821
Clarkson P A 1990 *Europe. J. Appl. Math.* **1** 279
Clarkson P A 1992 *Nonlinearity* **5** 453
Clarkson P A and Hood S 1993 *J. Phys. A: Math. Gen.* **26** 133
Clarkson P A and Hood S 1994 *J. Math. Phys.* **35** 255
Clarkson P A and Winternitz P, Clarkson P A, Clarkson A and Mansfield E L 1994 *Physica* **70D** 250
Clarkson P A 1995 *Chaos, Solitons & Fractals* **5** 2261
[5] Lou S-y 1990 *J. Phys. A: Math. Gen.* **23** L649
Lou S-y 1990 *Phys. Lett.* **151A** 133
Lou S-y 1991 *Sci. Sci. China* A **34** 1098
Lou S-y 1992 *J. Math. Phys.* **33** 4300
Lou S-y 1995 *Math. Methods in the Appl. Sci.* **18** 789
Lou S-y and Ni G-j 1991 *Commun. Theor. Phys.* **15** 4655
Lou S-y, Ruan H-y, Chen D-f and Chen W-z 1991 *J. Phys. A: Math. Gen.* **24** 1455
Lou S-y and Ruan H-y 1993 *J. Phys. A: Math. Gen.* **26** 4679
[6] Cao C W 1990 *Sci. China* A **33** 528
Cao C W and Geng X G 1990 *J. Phys. A: Math. Gen.* **23** 4117
Cao C W and Geng X G 1991 *J. Math. Phys.* **32** 2323
[7] Zeng Y B and Li Y S 1989 *J. Math. Phys.* **30** 1617
Zeng Y B and Li Y S 1990 *J. Phys. A: Math. Gen.* **23** L89
Zeng Y B 1991 *Phys. Lett.* **160A** 541
[8] Cheng Y and Li Y S 1991 *Phys. Lett.* **157A** 22
Cheng Y and Li Y S 1992 *J. Phys. A: Math. Gen.* **25** 419
Konopenlcnenko B, Sidorenko J and Strampp W 1991 *Phys. Lett.* **157A** 17
[9] Miura R M 1968 *J. Math. Phys.* **9** 1022
Miura R M 1968 *J. Math. Phys.* **9** 1024
[10] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
Nucci M C 1989 *J. Phys. A: Math. Gen.* **22** 2897
[11] Lou S-y 1993 *Phys. Lett.* **302B** 261
Lou S-y 1994 *J. Ningbo Normal College* **16** (5/6) 43

Lou S-y 1993 *Int. J. Mod. Phys. (Proc. Suppl.)* A **3** 531
Lou S-y 1994 *J. Math. Phys.* **35** 2336
Lou S-y 1994 *J. Math. Phys.* **35** 2390
[12] Lou S-y 1994 *Phys. Lett.* **187A** 239
Lou S-y 1997 *Phys. Scr.* at press
[13] Lou S-y 1993 *Phys. Lett.* **175A** 23
[14] McCoy B M and Perk J H H 1987 *Nucl. Phys.* B **285** 279
[15] Lou S-y and Ni G-j 1989 *Phys. Lett.* **140A** 33
Lou S-y and Ni G-j 1989 *J. Math. Phys.* **30** 1614
Lou S-y, Huang G-x and Ni G-j 1990 *Phys. Lett.* **146A** 45